ISRAEL JOURNAL OF MATHEMATICS, Vol. 69, No. 3, 1990

A REMARK ON HYPERCONTRACTIVITY AND THE CONCENTRATION OF MEASURE PHENOMENON IN A COMPACT RIEMANNIAN MANIFOLD

BY

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ABSTRACT

Following closely the Laplace operator method of M. Gromov and V. D. Milman (1983), we slightly improve their general estimate of the concentration function $\alpha(X;\varepsilon)$ of a compact Riemannian manifold X by showing that

$$\alpha(X;\varepsilon) \leqq C\exp(-c\varepsilon\sqrt{\rho_0}\log(1+\varepsilon\sqrt{\rho_0}))$$

for numerical constants C, $c>0$, where $\rho_0>0$ is the hypercontractive constant of the Laplacian on X.

Let (X, d) be a compact metric space and let μ be a probability measure on X equipped with its Borel σ-algebra. The concentration function $\alpha(X;\varepsilon) = \alpha((X, d, \mu);\varepsilon)$, $\varepsilon>0$, of the metric space (X, d) with respect to the probability measure μ has been introduced by M. Gromov and V. D. Milman [G–M] (see also [M–S], [Mi]) as

$$\alpha(X;\varepsilon) = 1 - \inf\{\mu(A_\varepsilon);\ A \text{ Borel subset of } X,\ \mu(A) \geqq \tfrac{1}{2}\}$$

where $A_\varepsilon = \{x \in X;\ d(x, A) \leqq \varepsilon\}$. In terms of functions, if f is Lipschitzian on (X, d) with Lipschitz norm $\|f\|_{\mathrm{Lip}}$, and if M_f is a Lévy mean or median of f for μ, i.e. $\mu(f \geqq M_f) \geqq \frac{1}{2}$, $\mu(f \leqq M_f) \geqq \frac{1}{2}$, then, for all $\varepsilon > 0$,

$$\mu(|f - M_f| \leqq \varepsilon\,\|f\|_{\mathrm{Lip}}) \geqq 1 - 2\alpha(X;\varepsilon),$$

so that f is *concentrated* around its median at a rate given by $\alpha(X;\varepsilon)$.

Received October 11, 1989

As investigated by the preceding authors, for a number of families (X, d, μ), the concentration function $\alpha(X;\varepsilon)$ turns out to be extremely small when ε becomes large (the range of interest for the values of ε being of course up to the diameter of X). In particular, many of them happen to have a *normal* concentration function in the sense that, for all $\varepsilon > 0$,

$$\alpha(X;\varepsilon) \leqq C\exp(-c\varepsilon^2)$$

for positive constants C, c (possibly depending on (X, d, μ)). For example, if X is the Euclidean sphere S_r^n of radius r and dimension n ($\geqq 2$) equipped with the geodesic distance d and the rotation invariant probability measure σ_r^n, the isoperimetric inequality on S_r^n (see e.g. [M–S]) implies that

$$\alpha(S_r^n;\varepsilon) \leqq \sqrt{\frac{\pi}{8}}\exp\left(-\frac{n-1}{2r^2}\varepsilon^2\right). \tag{1}$$

The various methods used so far to show that a concentration function is small are described in [G–M], [M–S], [Mi]. Among them, let us mention especially, as we have just seen, isoperimetric inequalities (from which the concentration idea was drawn), martingale and probabilistic inequalities, Laplace operator techniques, etc.

This note is concerned with the Laplace operator method on compact Riemannian manifolds. Let thus X denote a compact connected Riemannian manifold of dimension n ($\geqq 2$) with Riemannian metric d and normalized volume element μ.

Under geometric assumptions on the curvature, deep isoperimetric theorems yield a normal concentration function of X. Denote by $R(X) = \inf_\tau \mathrm{Ric}(\tau,\tau)$ the infimum of the Ricci curvature of X, where Ric is the Ricci tensor and the infimum runs over all unit tangent vectors (see [B–G–M]). Assume that $R(X) > 0$ and let r be such that $R(X) = R(S_r^n) = (n-1)/r^2$. Then, the Lévy–Gromov isoperimetric inequality ([Gr], [M–S]) expresses that for all Borel sets A in X, and all $\varepsilon > 0$,

$$\mu(A_\varepsilon) \geqq \sigma_r^n(B_\varepsilon)$$

where B is a ball (cap) of the Euclidean sphere S_r^n such that $\mu(A) = \sigma_r^n(B)$. This isoperimetric inequality, which extends the one on the sphere, yields as in (1), by an appropriate estimate of the measure of a cap, the concentration

$$\alpha(X;\varepsilon) \leqq \sqrt{\frac{\pi}{8}}\exp\left(-R(X)\frac{\varepsilon^2}{2}\right). \tag{2}$$

[The Lévy–Gromov inequality has been further extended in [B–B–G] and [Ga] to the case of a non-necessarily positive lower bound on the Ricci curvature. Besides improvements involving the diameter $d(X)$ of X on the case $R(X) > 0$ itself, one deduces again normal concentration. More precisely, it follows from [Ga], Théorème 6.16, that, for all $\varepsilon > 0$,

$$\alpha(X; \varepsilon) \leqq \sqrt{\frac{\pi}{8}} \exp\left(-\frac{(n-1)}{r_n(X)^2}\frac{\varepsilon^2}{2}\right)$$

where, when $R(X) = 0$,

$$r_n(X) = \frac{nd(X)}{2(\omega_n/2)^{1/n}}$$

and $\omega_n = \mathrm{vol}(S^n)/\mathrm{vol}(S^{n-1}) = 2\int_0^{\pi/2} \cos^{n-1} t\, dt$, and when $-\infty < R(X) < 0$, $r_n(X)$ is such that

$$\frac{n-1}{r_n(X)^2} = -R(X)\left(\min\left(\frac{\omega_n}{a_n(X)}, \left(\frac{\omega_n}{a_n(X)}\right)^{1/n}\right)\right)^2$$

where

$$a_n(X) = \int_0^{d(X)(-R(X)/(n-1))^{1/2}} (\mathrm{ch}\, 2t)^{(n-1)/2} dt.]$$

What can be said on the concentration function $\alpha(X; \varepsilon)$ of a compact Riemannian manifold X when no lower bound on the Ricci curvature is available? As shown in [G–M], the first non-trivial eigenvalue of the Laplace operator may be used to this aim. Let Δ be the Laplace–Beltrami operator on X. $-\Delta$ has a discrete spectrum consisting of the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. In particular, the minimax principle characterizes the first non-trivial eigenvalue $\lambda_1 = \lambda_1(X)$ of X as the largest constant in the Poincaré inequality

$$(3) \qquad \lambda_1 \int |f - \textstyle\int f d\mu|^2 d\mu \leqq \langle -\Delta f, f \rangle = \int |\nabla f|^2 d\mu$$

which holds for all smooth enough functions f on X. Let then $\delta > 0$ and A, B be two Borel subsets of X with positive measures $a = \mu(A)$, $b = \mu(B)$, at a distance larger than δ (i.e. $d(x, y) \geqq \delta$ whenever $x \in A$, $y \in B$). Applying the preceding Poincaré inequality to the function

$$f(x) = \frac{1}{a} - \frac{1}{\delta}\left(\frac{1}{a} + \frac{1}{b}\right) \min(d(x, A), \delta),$$

and iterating appropriately the resulting equation, M. Gromov and V. D. Milman [G–M] showed that the concentration function of X can always be estimated using $\lambda_1 = \lambda_1(X)$ as

$$\alpha(X;\varepsilon) \leqq C\exp(-c\sqrt{\lambda_1}\,\varepsilon) \tag{4}$$

(with $C = \frac{3}{4}$, $c = \log(\frac{3}{2})$). While this more general result does not allow one to recapture the estimates in case of a lower bound on the Ricci curvature (for example $\lambda_1(X) \geqq \lambda_1(S_r^n) = n/r^2$ where r is such that $R(X) = R(S_r^n) = (n-1)/r^2$, with equality only if X is isometric to S_r^n, cf. [B–G–M]), the discussion in [Mi] however indicates that it is essentially of the best possible type in general.

The purpose of this note is to slightly improve the preceding *general* result by use of another geometric invariant of the Laplace operator, namely its hypercontractive constant. O. S. Rothaus [Ro1] showed that for the Laplace operator Δ on the compact Riemannian manifold X, there exists $\rho > 0$ such that whenever $1 < p < q < \infty$ and $t > 0$ satisfy Nelson's hypercontractive condition [Ne]

$$e^{\rho t} \geqq \left(\frac{q-1}{p-1}\right)^{1/2},$$

then

$$\| e^{-t\Delta} \|_{p,q} \leqq 1$$

where $\| \cdot \|_{p,q}$ denotes the operator norm from $L^p(d\mu)$ to $L^q(d\mu)$. The best possible value $\rho_0 = \rho_0(X) > 0$ for this property to hold is called the *hypercontractive* constant of X (or of Δ on X).

In the spirit of the connection between λ_1 and Poincaré's inequality, the preceding hypercontractive estimate may be expressed equivalently, as was shown by L. Gross [Gro], as a *logarithmic Sobolev inequality*. We have namely that, for any smooth enough function f on X,

$$\rho_0 \int f^2 \log\left(\frac{|f|}{\| f \|_2}\right) d\mu \leqq \langle -\Delta f, f \rangle = \int |\nabla f|^2 d\mu \tag{5}$$

where $\| \cdot \|_2$ is the L^2 norm on X. It is easily seen and observed in [Ro1] that $\rho_0 \leqq \lambda_1$. Indeed, given f smooth enough with mean zero, apply (5) to $1 + \varepsilon f$ and let then ε tend to 0 to get Poincaré's inequality (3) with ρ_0; hence $\rho_0 \leqq \lambda_1$.

As for the first eigenvalue of $-\Delta$, it is of interest to find handy and sharp lower bounds of its hypercontractive constant ρ_0. For spheres, it is shown in [M–W] that $\rho_0(S_r^n) = \lambda_1(S_r^n) = n/r^2$, while in case the Ricci curvature is positive, D. Bakry and M. Emery [B–E] proved, using probabilistic methods, that $\rho_0(X) \geqq \rho_0(S_r^n)$ where, as usual, r is such that $R(X) = R(S_r^n) = (n-1)/r^2$. (Although this has not yet been explicitly described, the latter should be no surprise from the Lévy–Gromov isoperimetric inequality.) Since as a consequence of the Bochner–Lichnerowicz–Weitzenböck formula (see [B–G–M]) we have that

$$\tfrac{1}{2}\Delta(|\nabla f|^2) - \nabla f \circ \nabla(\Delta f) \geqq R(X)|\nabla f|^2 + \frac{1}{n}(\Delta f)^2,$$

the preceding lower bound on $\rho_0(X)$ has been improved by O. S. Rothaus [Ro2] (including cases for which $R(X) \leqq 0$) into

$$\rho_0(X) \geqq \frac{1 - 1/n}{(1 + 1/n)^2} R(X) + \frac{4/n}{(1 + 1/n)^2} \lambda_1(X).$$

In other words, when $R(X) > 0$ and $r^2 = (n-1)/R(X)$,

$$\rho_0(X) \geqq \frac{(1 - 1/n)^2}{(1 + 1/n)^2} \rho_0(S_r^n) + \left(1 - \frac{(1 - 1/n)^2}{(1 + 1/n)^2}\right) \lambda_1(X).$$

It follows in particular that if $\rho_0(X) = \rho_0(S_r^n)$, then necessarily $\lambda_1(X) \leqq \rho_0(S_r^n)$, so that $\lambda_1(X) = \lambda_1(S_r^n)$ and X is also isometric to S_r^n in this case. I am grateful to M. Emery for this observation that he noticed in the process of the work [B–E] as a consequence of some of the computations developed there.

The result of this note is an improvement of the general estimate (4) of $\alpha(X;\varepsilon)$ in terms of the hypercontractive constant of X.

Theorem. *Let X be a compact connected Riemannian manifold with hypercontractive constant $\rho_0 = \rho_0(X) > 0$. Then, for numerical constants $C, c > 0$,*

$$\alpha(X;\varepsilon) \leqq C \exp(-c\varepsilon\sqrt{\rho_0}\log(1 + \varepsilon\sqrt{\rho_0}))$$

for all $\varepsilon > 0$.

The proof of this theorem just mimics the argument of M. Gromov and V. D. Milman, applying the logarithmic Sobolev inequality (5) to the (somewhat more convenient) function

$$f(x) = \frac{1}{a} + \frac{1}{\delta}\left(\frac{1}{b} - \frac{1}{a}\right)\min(d(x, A), \delta)$$

where $a = \mu(A) \geqq \mu(B) = b$ and A, B are Borel subsets at a distance larger than $\delta > 0$.

It seems that it has still to be understood what characterizes normal concentration of compact Riemannian manifolds.

Proof of the Theorem. Let therefore A and B be Borel subsets of X such that $d(x, y) \geqq \delta$ for all $x \in A$ and $y \in B$, and for some $\delta > 0$ to be specified. We assume that $a = \mu(A) \geqq \frac{1}{2}$, hence in particular, $b = \mu(B) \leqq 1 - a \leqq \frac{1}{2} \leqq a$. As announced, consider f given by

$$f(x) = \frac{1}{a} + \frac{1}{\delta}\left(\frac{1}{b} - \frac{1}{a}\right) \min(d(x, A), \delta)$$

(so that $f = 1/a$ on A, $f = 1/b$ on B and, everywhere, $1 \leqq 1/a \leqq f \leqq 1/b$). Apply to this function the logarithmic Sobolev inequality (5) in the following equivalent formulation (homogeneity):

$$\int f^2 \log f^2 d\mu \leqq \frac{2}{\rho_0} \int |\nabla f|^2 d\mu + \left(\int f^2 d\mu\right) \log\left(\int f^2 d\mu\right).$$

We need simply evaluate appropriately the various terms of this inequality for the preceding function f. First, by convexity of $x \log^+ x$ and Jensen's inequality, $(\cdot)^c$ denoting complementation,

$$\begin{aligned}
\int f^2 \log f^2 d\mu &= \frac{1}{a} \log \frac{1}{a^2} + \frac{1}{b} \log \frac{1}{b^2} + \int_{(A \cup B)^c} f^2 \log f^2 d\mu \\
&\geqq \frac{1}{b} \log \frac{1}{b^2} + \int_{(A \cup B)^c} f^2 d\mu \log\left(\frac{1}{\mu((A \cup B)^c)} \int_{(A \cup B)^c} f^2 d\mu\right) \\
&= \frac{1}{b} \log \frac{1}{b^2} + \int_{(A \cup B)^c} f^2 d\mu \log\left(\frac{1}{1 - a - b} \int_{(A \cup B)^c} f^2 d\mu\right).
\end{aligned}$$

On the other hand, clearly,

$$\int |\nabla f|^2 d\mu \leqq \frac{1}{\delta^2}\left(\frac{1}{b} - \frac{1}{a}\right)^2 (1 - a - b) \leqq \frac{1}{\delta^2 b^2}(1 - a - b)$$

while

$$\int f^2 d\mu = \frac{1}{a} + \frac{1}{b} + \int_{(A \cup B)^c} f^2 d\mu.$$

We have thus the inequality

$$
\begin{aligned}
&\frac{1}{b}\log\frac{1}{b^2}+\int_{(A\cup B)^c} f^2 d\mu \log\left(\frac{1}{1-a-b}\int_{(A\cup B)^c} f^2 d\mu\right) \\
(6)\quad &\leqq \frac{2}{\rho_0\delta^2 b^2}(1-a-b)+\left(\frac{1}{a}+\frac{1}{b}+\int_{(A\cup B)^c} f^2 d\mu\right)\log\left(\frac{1}{a}+\frac{1}{b}+\int_{(A\cup B)^c} f^2 d\mu\right).
\end{aligned}
$$

On the basis of this inequality we distinguish between two cases. If

$$
\log\left(\frac{1}{a}+\frac{1}{b}+\int_{(A\cup B)^c} f^2 d\mu\right)\leqq \log\left(\frac{1}{1-a-b}\int_{(A\cup B)^c} f^2 d\mu\right),
$$

that is, if

$$
\int_{(A\cup B)^c} f^2 d\mu \geqq \frac{1-a-b}{ab},
$$

then (6) implies that

$$
\frac{1}{b}\log\frac{1}{b^2}\leqq \frac{2}{\rho_0\delta^2 b^2}(1-a-b)+\left(\frac{1}{a}+\frac{1}{b}\right)\log\left(\frac{1}{a}+\frac{1}{b}+\int_{(A\cup B)^c} f^2 d\mu\right)
$$

and since f is always less than $1/b$ and $b \leqq 1-a$, we get that

$$
\begin{aligned}
&\frac{1}{b}\log\frac{1}{b^2}\leqq \frac{2}{\rho_0\delta^2 b^2}(1-a-b)+\left(\frac{1}{a}+\frac{1}{b}\right)\log\left(\frac{1}{a}+\frac{1-a}{b^2}\right) \\
(7)\quad &\leqq \frac{2}{\rho_0\delta^2 b^2}(1-a-b)+\left(\frac{1}{a}+\frac{1}{b}\right)\log\left(\frac{1-a}{ab^2}\right).
\end{aligned}
$$

On the other hand, if

$$
\int_{(A\cup B)^c} f^2 d\mu \leqq \frac{1-a-b}{ab},
$$

then, clearly,

$$
\frac{1}{a}+\frac{1}{b}+\int_{(A\cup B)^c} f^2 d\mu \leqq \frac{1}{ab}
$$

so that (6) reads in this case as

$$
(8)\qquad \frac{1}{b}\log\frac{1}{b^2}\leqq \frac{2}{\rho_0\delta^2 b^2}(1-a-b)+\frac{1}{ab}\log\frac{1}{ab}.
$$

Let us now fix the choice δ and take $\delta = 1/\sqrt{\rho_0}$. Let us assume further to begin with (and actually for simplicity) that $a \geqq \frac{3}{4}$ (and thus $b \leqq 1 - a \leqq \frac{1}{4}$). In case of (7), we have that

$$\frac{1}{b}\log\frac{1}{b^2} \leqq \frac{2}{b^2}(1-a) - \frac{2}{b} + \frac{1}{b}\log\left(\frac{1-a}{b^2}\right) + \frac{1}{a}\log\left(\frac{1-a}{b^2}\right) + \left(\frac{1}{a}+\frac{1}{b}\right)\log\frac{1}{a}.$$

Therefore, since $a \geqq \frac{3}{4}$,

$$\begin{aligned}\frac{1}{b}\log\left(\frac{1}{1-a}\right) &\leqq \frac{2}{b^2}(1-a) - \frac{2}{b} + \frac{4}{3}\log\frac{1}{4b^2} + \left(\frac{4}{3}+\frac{1}{b}\right)\log\frac{4}{3} \\ &\leqq \frac{2}{b^2}(1-a)\end{aligned}$$

and thus

$$b \leqq 2(1-a)\left[\log\left(\frac{1}{1-a}\right)\right]^{-1}.$$

In case of (8), and always for $a \geqq \frac{3}{4}$,

$$\begin{aligned}\frac{1}{ab}\log\frac{1}{ab} &= \frac{1}{ab}\log\frac{1}{a} + \frac{1}{ab}\log\frac{1}{b} \\ &\leqq \left(\frac{4}{3}\log\frac{4}{3}\right)\frac{1}{b} + \frac{4}{3}\cdot\frac{1}{b}\log\frac{1}{b}\end{aligned}$$

so that

$$\frac{2}{3}\cdot\frac{1}{b}\log\frac{1}{b} \leqq \frac{2}{b^2}(1-a).$$

Since $b \leqq 1 - a$, we have thus obtained that, in any case,

$$b \leqq 3(1-a)\left[\log\left(\frac{1}{1-a}\right)\right]^{-1}. \tag{9}$$

Further, this inequality is also satisfied for $\frac{1}{2} \leqq a < \frac{3}{4}$.

We can now conclude the proof of the theorem. Let A satisfy $\mu(A) \geqq \frac{1}{2}$. Set $\delta = 1/\sqrt{\rho_0}$ and define, for $n \geqq 1$, $A(n) = A_{(n-1)\delta}$ ($A(1) = A$), and $B(n) = (A(n)_\delta)^c = (A_{n\delta})^c$. Set further $b_n = \mu(B(n))$ and $b_0 = 1 - \mu(A)$. We can apply (9) to the couples $(A(n), B(n))$ for each n to get that

$$b_{n+1} \leqq 3b_n\left(\log\frac{1}{b_n}\right)^{-1}, \qquad n \geqq 0. \tag{10}$$

We recall at this stage that, by (4), and since $\lambda_1 \geqq \rho_0$, for all n,

$$b_n \leqq \frac{3}{4} \exp\left(- n\delta \sqrt{\rho_0} \log \frac{3}{2}\right) \leqq \exp\left(- \frac{n}{3}\right).$$

Hence, for all $n \geqq 1$,

$$b_{n+1} \leqq 9 b_n / n$$

($b_1 \leqq b_0 \leqq \frac{1}{2}$). Iteration of this inequality yields

$$b_{n+1} \leqq \frac{9^n}{n!} b_1 \leqq \left(\frac{9e}{n}\right)^n .$$

It follows that for some numerical constants $C, c > 0$,

$$b_n \leqq C \exp(- cn \log(n + 1))$$

for all $n \geqq 0$. [For example, but this is not sharp, one can take $C = 10^3$, $c = \frac{1}{4}$.] Let now $\varepsilon > 0$; take $n \geqq 0$ with $n\delta \leqq \varepsilon \leqq (n + 1)\delta$. Thus

$$\begin{aligned} 1 - \mu(A_\varepsilon) \leqq b_n &\leqq C \exp(- cn \log(n + 1)) \\ &\leqq C \exp\left(- \frac{c\varepsilon}{4\delta} \log\left(1 + \frac{\varepsilon}{\delta}\right)\right) \end{aligned}$$

as least if $n \geqq 1$. If $n = 0$,

$$1 - \mu(A_\varepsilon) \leqq \frac{1}{2} \leqq C \exp(- c \log 2) \leqq C \exp\left(- c \frac{\varepsilon}{\delta} \log\left(1 + \frac{\varepsilon}{\delta}\right)\right)$$

for appropriate $C, c > 0$ ($C \geqq 1$, $c \leqq 1$). The proof is complete.

References

[B–B–G] P. Bérard, G. Besson and S. Gallot, *Sur une inégalité isopérimétrique qui généralise celle de Paul Lévy-Gromov*, Invent. Math. **80** (1985), 295–308.

[B–E] D. Bakry and M. Emery, *Diffusions hypercontractives*, Séminaire de Probabilités XIX, Lecture Notes in Math. **1123**, Springer-Verlag, Berlin, 1985, pp. 175–206.

[B–G–M] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Math. **194**, Springer-Verlag, Berlin, 1971.

[Ga] S. Gallot, *Inégalités isopérimétriques et analytiques sur les variétés riemanniennes*, On the Geometry of differentiable manifolds, Astérisque **163–164** (1988), 31–91.

[Gr] M. Gromov, *Paul Lévy's isoperimetric inequality*, Publication de l'I.H.E.S., 1980.

[G–M] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality*, Am. J. Math. **105** (1983), 843–854.

[Gro] L. Gross, *Logarithmic Sobolev inequalities*, Am. J. Math. **97** (1975), 1061–1083.

[Mi] V. D. Milman, *The heritage of P. Lévy in geometrical functional analysis*, Colloque Paul Lévy sur les processus stochastiques, Astérisque **157-158** (1988), 273–301.

[M–S] V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer-Verlag, Berlin, 1986.

[M–W] C. Mueller and F. Weissler, *Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere*, J. Funct. Anal. **48** (1982), 252–283.

[Ne] E. Nelson, *The free Markov field*, J. Funct. Anal. **12** (1973), 211–227.

[Ro1] O. S. Rothaus, *Diffusion on compact Riemannian manifolds and logarithmic Sobolev inequalities*, J. Funct. Anal. **42** (1981), 102–109.

[Ro2] O. S. Rothaus, *Hypercontractivity and the Bakry–Emery criterion for compact Lie groups*, J. Funct. Anal. **65** (1986), 358–367.